EMBEDDING PROPER ULTRAMETRIC SPACES INTO ℓ_p AND ITS APPLICATION TO NONLINEAR DVORETZKY'S THEOREM

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Abstract. We prove that every proper ultrametric space isometrically embeds into ℓ_p for any $p \geq 1$. As an application we discuss an ℓ_p-version of nonlinear Dvoretzky's theorem.

1. Introduction

Recall that a metric space (X, ρ) is called an *ultrametric space* if for every $x, y, z \in X$ we have $\rho(x, y) \leq \max\{\rho(x, z), \rho(z, y)\}$. Such spaces naturally appear and have applications in various areas such as number theory, p-adic analysis, and computer science (see [9], [10], [16, 17]).

Let us briefly review several results with respect to isometric embedding of ultrametric spaces. Timan and Vestfrid [21, 22] proved that any separable ultrametric space embed isometrically into ℓ_2. Vestfrid [24] later proved that the result is also true if one replace ℓ_2 by ℓ_1 and c_0 by constructing a universal ultrametric space for the class of separable ultrametric space and using its property. Vestfrid [23] also proved that a certain class of countable ultrametric spaces embed isometrically into ℓ_p for $p \geq 1$. Lemin [10] proved that any separable ultrametric space embed isometrically into the Lebesgue space. He also raised a problem whether any separable ultrametric space embed isometrically into any infinite dimensional Banach space. Motivated by Lemin's problem, Shkarin [20] proved that every finite ultrametric space embeds into every infinite dimensional Banach space. From these results ultrametric spaces have attracted much attention in embedding theory.

In this paper we tackle Lemin's problem in the case where the target Banach space is ℓ_p. It is already well-known that every separable ultrametric space embeds isometrically into the function space L_p for any $p \geq 1$. In fact, it follows from Timan and Vestfrid's result mentioned above and the fact that ℓ_2 embeds isometrically into L_p. Since ℓ_2 does not embed bi-Lipschitzly into ℓ_p for any $p \neq 2$ ([1, Corollary 2.1.6]), embedding separable ultrametric spaces into ℓ_p is left as a problem. Our main theorem is the following: Recall that a metric space is *proper* if every closed ball in X is compact.

Theorem 1.1. *Every proper ultrametric space isometrically embeds into ℓ_p for any $p \geq 1$.*

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The case of general separable ultrametric spaces remains open. A similar method of the proof of Theorem 1.1 also implies an isometric embedding into c_0 (see Remark 2.3). Our construction of isometric embeddings into ℓ_1, ℓ_2, and c_0 is different from the one by [21, 22], [23, 24] in the case of proper ultrametric spaces.

As an application of Theorem 1.1 we obtain an ℓ_p-version of nonlinear Dvoretzky's theorem, see Section 3.

2. Proof of the main theorem

We use some basic facts of compact ultrametric spaces (see [8], [12, Section 2]). Let (X,ρ) be a compact ultrametric space and put $r_0 := \operatorname{diam} X$. Consider the relation $\sim_0$ on X given by $x \sim_0 y \Longleftrightarrow \rho(x,y) < r_0$. Since ρ is ultrametric $\sim_0$ is an equivalence relation on X. The compactness of X implies that each equivalence class is a closed ball of radius strictly less than r_0 (see [12, Section 2]). Since the distance between two distinct equivalence classes is exactly r_0 and X is totally bounded, there are only finitely many equivalence classes, say, $\{B_1, \cdots, B_{k_1}\}$, where each B_i is a closed ball of radius $r_i = \operatorname{diam} B_i < r_0$. Note that for any $x \in B_i$ and $y \in B_j$ $(i \neq j)$ we have $\rho(x,y) = r_0$. For each i we choose $x_i \in B_i$ and fix it. As above for each $i_1 = 1, \cdots, k_1$ we consider the equivalence relation $\sim_{i_1}$ on B_{i_1} given by $x \sim_{i_1} y \Longleftrightarrow \rho(x,y) < r_{i_1}$. Then we can divide X_{i_1} into finitely many equivalence classes, i.e., $B_{i_1} = \amalg_{i_2=1}^{k(i_1)} B_{i_1 i_2}$, where $B_{i_1 i_2}$ is a closed ball of radius $r_{i_1 i_2} = \operatorname{diam} B_{i_1 i_2} < r_{i_1}$. We may assume that $x_{i_1} \in B_{i_1 1}$. For each i_1, i_2, we choose a point $x_{i_1 i_2} \in B_{i_1 i_2}$ so that $x_{i_1 1} = x_{i_1}$ and we fix $x_{i_1 i_2}$. Repeatedly we get a sequence $\mathcal{P}_k = \{B_{i_1 \cdots i_k}\}_{i_1, \cdots, i_k}$ of partitions of X satisfying the following:

(1) Each $B_{i_1 \cdots i_k}$ is a closed ball of radius $r_{i_1 \cdots i_k} = \operatorname{diam} B_{i_1 \cdots i_k}$.
(2) If $r_{i_1 \cdots i_k} \neq 0$, then $r_{i_1 \cdots i_k} > r_{i_1 \cdots i_{k+1}}$.
(3) $B_{i_1 \cdots i_{k-1}} = \amalg_{i_k} B_{i_1 \cdots i_{k-1} i_k}$.

For each $i_1, \cdots, i_k$ we choose $x_{i_1 \cdots i_k} \in B_{i_1 \cdots i_k}$ so that $x_{i_1 \cdots i_k 1 \cdots 1} = x_{i_1 \cdots i_k}$. The compactness of X yields the following:

Lemma 2.1 (cf. [12, Section 2])**.** $\lim_{k\to\infty} \max_{i_1, \cdots, i_k} r_{i_1 \cdots i_k} = 0$.

In particular, $\bigcup_{k=1}^{\infty} \{x_{i_1 \cdots i_k}\}_{i_1, \cdots, i_k}$ is a countable dense subset of X.

Lemma 2.2 (cf. [12, Section 2])**.** *For every closed ball B in X, there exist k and $B_{i_1 \cdots i_k} \in \mathcal{P}_k$ such that $B = B_{i_1 \cdots i_k}$.*

Proof of Theorem 1.1. We first prove the theorem for compact ultrametric spaces. Let (X,ρ) be a compact ultrametric space and let $\mathcal{P}_k = \{B_{i_1 \cdots i_k}\}_{i_1, \cdots, i_k}$, $r_{i_1 \cdots i_k}$, and $x_{i_1 \cdots i_k}$ as above. Put $N_k := \# \mathcal{P}_k$. We consider each coordinate of an element of $\ell_p^{N_k}$ is indexed by $(i_1, \cdots, i_k)$. We define a map $f_k : \{x_{i_1 \cdots i_k}\}_{i_1, \cdots, i_k} \to \ell_p^{N_k}$ as follows: $(f_k(x_{i_1 \cdots i_k}))_{(j_1, \cdots, j_k)} := 0$ if $(j_1, \cdots, j_k) \neq (i_1, \cdots, i_k)$ and

$$(f_1(x_{i_1}))_{i_1} := \frac{(r_0^p - r_{i_1}^p)^{\frac{1}{p}}}{2^{\frac{1}{p}}} \text{ and } (f_k(x_{i_1 \cdots i_k}))_{(i_1, \cdots, i_k)} := \frac{(r_{i_1 \cdots i_{k-1}}^p - r_{i_1 \cdots i_k}^p)^{\frac{1}{p}}}{2^{\frac{1}{p}}} \text{ if } k \geq 2.$$

Note that $f_k(x_{i_1 \cdots i_k}) \perp f_k(x_{j_1 \cdots j_k})$ for two distinct $(i_1, \cdots, i_k)$, $(j_1, \cdots, j_k)$.

We define a map $f : \bigcup_{k=1}^{\infty} \{x_{i_1 \cdots i_k}\}_{i_1, \cdots, i_k} \to \ell_p$ as follows. For each $x_{i_1 \cdots i_k}$, putting $i_m := 1$ for $m > k$, we define

$$f(x_{i_1 i_2 \cdots i_k}) := (f_1(x_{i_1}), f_2(x_{i_1 i_2}), \cdots, f_m(x_{i_1 \cdots i_m}), \cdots).$$

The right-hand side in the above definition is actually the element of ℓ_p since

$$\sum_{m=1}^{\infty} \|f_m(x_{i_1\cdots i_m})\|_p^p = \sum_{m=1}^{\infty} \frac{r_{i_1\cdots i_{m-1}}^p - r_{i_1\cdots i_m}^p}{2} = \frac{r_0^p}{2} < +\infty$$

by Lemma 2.1. Note that f is well-defined in the sense that $f(x_{i_1\cdots i_k 1\cdots 1}) = f(x_{i_1\cdots i_k})$.

We shall prove that f is an isometric embedding. Since $\bigcup_{k=1}^{\infty}\{x_{i_1\cdots i_k}\}_{i_1,\cdots,i_k}$ is dense in X this implies the theorem. Taking two distinct elements $x_{i_1\cdots i_k}$ and $x_{j_1\cdots j_l}$ we may assume that $k \le l$. Put $i_m := 1$ for $m > k$. Then we have $(i_1, \cdots, i_l) \ne (j_1, \cdots, j_l)$. Letting

$$n := \min\{m \le l \mid i_m \ne j_m\}$$

we get $\rho(x_{i_1\cdots i_k}, x_{j_1\cdots j_l}) = \operatorname{diam} B_{i_1\cdots i_{n-1}} = r_{i_1\cdots i_{n-1}}$ if $n \ge 2$ and $\rho(x_{i_1\cdots i_k}, x_{j_1\cdots j_l}) = r_0$ if $n = 1$. Since $f_m(x_{i_1\cdots i_m}) = f_m(x_{j_1\cdots j_m})$ for $m < n$ and $f_m(x_{i_1\cdots i_m}) \perp f_m(x_{j_1\cdots j_m})$ for $m \ge n$,

$$\begin{aligned}
\|f(x_{i_1\cdots i_k}) - f(x_{j_1\cdots j_l})\|_p^p &= \sum_{m=n}^{\infty} \|f(x_{i_1\cdots i_m})\|_p^p + \sum_{m=n}^{\infty} \|f(x_{j_1\cdots j_m})\|_p^p \\
&= r_{i_1\cdots i_{n-1}}^p \\
&= \rho(x_{i_1\cdots i_k}, x_{j_1\cdots j_l})^p.
\end{aligned}$$

This completes the proof of the theorem for compact ultrametric spaces.

Let (X, ρ) be a proper ultrametric space and fix a point $x_0 \in X$. For any $r > 0$ we denote by $B(x_0, r)$ the closed ball of radius r centered at x_0. For any $R > 0$ let $f_1 : B(x_0, R) \to \ell_p$ be an isometric embedding constructed as in the above way. It suffices to prove that for any $R' > R$ we can construct an isometric embedding $f_2 : B(x_0, R') \to \ell_p$ as in the above way, which extends f_1 in the following sense: There exists an isometry $T : \ell_p \to \ell_p$ such that $T \circ f_2|_{B(x_0,R)} = f_1$. This is possible by the above construction. In fact, keep dividing $B(x_0, R')$ as in the above way. Then at finite steps we reach at $B(x_0, R)$ by Lemma 2.2 since $B(x_0, R')$ is compact. From the above construction we easily see the existence of f_2 and T. This completes the proof of the theorem. $\square$

Remark 2.3. A similar method of the above proof implies new isometric embeddings of proper ultrametric spaces into c_0. In fact, let us consider first the case of compact ultrametric spaces. Using the same notation as above, for each k we define $g_k : \{x_{i_1\cdots i_k}\}_{i_1,\cdots,i_k} \to \ell_\infty^{N_k}$ as follows: $(g_k(x_{i_1\cdots i_k}))_{(j_1,\cdots,j_k)} := 0$ if $(j_1, \cdots, j_k) \ne (i_1, \cdots, i_k)$ and

$$(g_1(x_{i_1}))_{i_1} := r_0 \text{ and } (g_k(x_{i_1\cdots i_k}))_{(i_1,\cdots,i_k)} := r_{i_1\cdots i_{k-1}} \text{ if } k \ge 2.$$

Then we define a map $g : \bigcup_{k=1}^{\infty}\{x_{i_1\cdots i_k}\}_{i_1,\cdots,i_k} \to c_0$ by

$$g(x_{i_1 i_2\cdots i_k}) := (g_1(x_{i_1}), g_2(x_{i_1 i_2}), \cdots, g_m(x_{i_1\cdots i_m}), \cdots),$$

where as in the above proof we put $i_m := 1$ for $m > k$. Note that the right-hand side of the above definition is in c_0 by Lemma 2.1. We can easily check that the map $g : \bigcup_{k=1}^{\infty}\{x_{i_1\cdots i_k}\}_{i_1,\cdots,i_k} \to c_0$ is an isometric embedding. As in the proof of Theorem 1.1 this construction also implies an isometric embedding from every proper ultrametric space into c_0.

3. ℓ_p-version of nonlinear Dvoretzky's theorem

In this section we apply Theorem 1.1 to obtain an ℓ_p-version of nonlinear Dvoretzky's theorem. Refer to [3], [5] for the case of finite metric spaces.

We say that a metric space X is *embedded with distortion* $D \geq 1$ in a metric space Y if there exist a map $f : X \to Y$ and a constant $r > 0$ such that

$$r\, d_X(x, y) \leq d_Y(f(x), f(y)) \leq Dr\, d_X(x, y) \text{ for all } x, y \in X.$$

Dvoretzky's theorem states that for every $\varepsilon > 0$, every n-dimensional normed space contains a $k(n, \varepsilon)$-dimensional subspace that embeds into a Hilbert space with distortion $1 + \varepsilon$ ([6]). This theorem was conjectured by Grothendieck ([7]). See [14] and [15], [19] for the estimate of $k(n, \varepsilon)$. Bourgain, Figiel, and Milman [4] first studied Dvoretzky's theorem in the nonlinear setting. They obtained that for every $\varepsilon > 0$, every finite metric space X contains a subset S of sufficiently large size which embeds into a Hilbert space with distortion $1 + \varepsilon$. See [2], [11], [18] for further investigation. Recently Mendel and Naor [12, 13] studied an another variant of nonlinear Dvoretzky's theorem, answering a question by T. Tao. For example they obtained the following: For a metric space X we denote by $\dim_H(X)$ the Hausdorff dimension of X.

Theorem 3.1 (cf. [13, Theorem 1.7])**.** *There exists a universal constant $c \in (0, \infty)$ such that for every $\varepsilon \in (0, \infty)$, every compact metric space X contains a closed subset $S \subseteq X$ that embeds with distortion $2 + \varepsilon$ in an ultrametric space, and*

$$\dim_H(S) \geq \frac{c\,\varepsilon}{\log(1/\varepsilon)} \dim_H(X).$$

Note that since every separable ultrametric space isometrically embed into ℓ_1, ℓ_2, and c_0 ([24]), the above S embeds into these spaces.

Applying Theorem 1.1 to Theorem 3.1 we obtain the following ℓ_p-version of nonlinear Dvoretzky's theorem:

Corollary 3.2. *There exists a universal constant $c \in (0, \infty)$ such that for every $\varepsilon \in (0, \infty)$, every compact metric space X contains a closed subset $S \subseteq X$ that embeds with distortion $2 + \varepsilon$ in ℓ_p, and*

$$\dim_H(S) \geq \frac{c\,\varepsilon}{\log(1/\varepsilon)} \dim_H(X).$$

Mendel and Naor also obtained the following impossibility result for distortion less than 2:

Theorem 3.3 (cf. [13, Theorem 1.8])**.** *For every $\alpha > 0$ there exists a compact metric space (X, d) of Hausdorff dimension α, such that if $S \subseteq X$ embeds into a Hilbert space with distortion strictly smaller than 2 then $\dim_H(S) = 0$.*

We shall consider an impossibility problem for the ℓ_p-version of nonlinear Dvoretzky's theorem.

In the proof of Theorem 3.3 Mendel and Naor used the following result: Let G be the random graph on n-vertices of the Erdös-Reyni model $G(n, 1/2)$, i.e., every edge is present independently with probability $1/2$. From G we construct a metric space W_n by assigning the distance between each two vertices of G by 1 if they are joined by an edge, and 2 if they are not joined by an edge. Then the obtained metric space W_n satisfies the following property ([2]). There exists $K \in (0, \infty)$ such

that for any $n \in \mathbb{N}$ there exists an n-point metric space W_n such that for every $\delta \in (0,1)$ any subset of W_n of size larger than $2\log_2 n + K(\delta^{-2}\log(2/\delta))^2$ must incur distortion at least $2-\delta$ when embedded into ℓ_2.

Bartal, Linial, Mendel, and Naor obtained a similar result for the same W_n when considering ℓ_p instead of ℓ_2 ([3]). Then Charikar and Karagiozova [5, Theorem 1.3] improved the result in [3]: For any $\delta \in (0,1)$ and $p \geq 1$, there is a constant $c(p,\delta)$ depending only on p and δ such that any subset of W_n of size larger than $c(p,\delta)\log n$ must incur distortion at least $2-\delta$ when embedded into ℓ_p.

Then applying this result to the proof in [13, Section 7.3] implies the following:

Proposition 3.4. *For every $p \geq 1$ and $\alpha > 0$, there exists a compact metric space (X,d) with $\dim_H(X,d) = \alpha$, such that if $S \subseteq X$ embeds into ℓ_p with distortion strictly smaller than 2 then $\dim_H(S) = 0$.*

The case of the distortion 2 remains open for any $p \geq 1$.

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